Encoding the Hamiltonian Cycle Problem into SAT Based on Vertex Elimination

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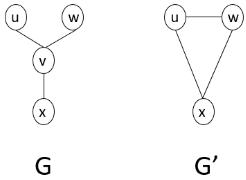
Introduction

- The Hamiltonian Cycle Problem (HCP)
  - Given a directed graph $G = (V, E)$, synthesize a subgraph $H_G = (H_V, H_E)$ which is cycle connecting each and every vertex in H_v exactly once.
    - Degree and no-sub-cycle constraints
- Existing SAT Encodings of HCP
  - Distance (position) encoding [MiniZinc, Zhou20]
  - Reachability (relative) encoding [Prestwich03]
  - Bijection encoding [HertelHU07]
  - Lazy (incremental) encodings [Dantzig54,SohBRBT14,Heule21]

Introduction (Cont.)

- Vertex elimination
  - Simplify a complex graph by removing certain vertices from the graph while preserving important properties.
- Vertex elimination encoding for directed graph acyclicity [RankoohR22]
  - Good for sparse graphs ☺, but expensive for dense graphs ☹
- A hybrid encoding based on vertex elimination and leaf elimination [ZhouWY23] ☺
- Can vertex elimination be used to encode HCP?



The HCP

- Given a directed graph $G = (V, E)$
- Synthesize a subgraph $H_G = (H_V, H_E)$
    - $H_V = \{v \mid v \in V,\ b_v = 1\}$
    - $H_E = \{(u, v) \mid (u, v) \in E,\ b_{uv} = 1\}$
- The `circuit` and `subcircuit` constraints are special cases.

The Distance Encoding for HCP

- Let k be the cardinality of H_V: $k = \sum_{v \in V} b_v$.
- For each vertex v, $s_v = 1$ iff v is the starting vertex.
- For each vertex v, d_v $(0 \leq d_v \leq n-1)$ is v's distance from the starting vertex in H_G.
- Degree constraints

For each $v \in V$:

$$k > 1 \wedge b_v \rightarrow \sum_{(u,v) \in E} b_{uv} = 1 \qquad \text{(D-1)}$$

$$k > 1 \wedge b_v \rightarrow \sum_{(v,w) \in E} b_{vw} = 1 \qquad \text{(D-2)}$$

The Distance Encoding for HCP (Cont.)

- Constraints on the starting vertex

$$k > 1 \rightarrow \sum_{v \in V} s_v = 1 \qquad \text{(D-3)}$$

For each $v \in V$:

$$s_v \rightarrow b_v \qquad \text{(D-4)}$$

$$s_v \rightarrow d_v = 0 \qquad \text{(D-5)}$$

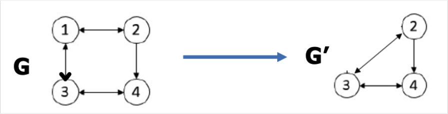
- Distance constraints

For each $(u, v) \in E$: $b_{uv} \wedge \neg s_v \rightarrow d_v = d_u + 1$ (D-6)

- Encoding size
  - $O(n^3)$ if unary encoding is used for distance variables
  - $O(n^2 \times log_2(n))$ if binary encoding is used for distance variables

Vertex Elimination Encoding for HCP

- Vertex elimination $G = (V, E) \rightarrow G' = (V', E')$



$$
\begin{aligned}
V' &= V \setminus \{v\} \\
E' &= E \setminus \{(u,v) \mid (u,v) \in E\} \\
&\quad \setminus \{(v,w) \mid (v,w) \in E\} \\
&\quad \cup \{(u,w) \mid u \in nbs^{-}(v),\ w \in nbs^{+}(v),\ u \neq w\}
\end{aligned}
$$

where

$$
\begin{aligned}
nbs^{-}(v) &= \{u \mid (u,v) \in E\} \\
nbs^{+}(v) &= \{w \mid (v,w) \in E\}.
\end{aligned}
$$

- For each vertex $u \in V'$: $b'_u = b_u$
- For each arc $(u, w) \in E' \cap E$:

  if $(u, v) \notin E$ or $(v, w) \notin E$, then $b_{uw} = b'_{uw}$

Vertex Elimination Encoding for HCP (Cont.)

- Degree constraints on the eliminated vertex v

$$k > 1 \wedge b_v \rightarrow \sum_{(u,v)\in E} b_{uv} = 1 \quad \text{(VE-1)}$$
$$k > 1 \wedge b_v \rightarrow \sum_{(v,w)\in E} b_{vw} = 1 \quad \text{(VE-2)}$$

- No cycle of size 2 involving the eliminated vertex v

For each $(u, v) \in E$, if $(v, u) \in E$:

$$k' > 1 \rightarrow \neg b_{uv} \vee \neg b_{vu} \quad \text{(VE-3)}$$

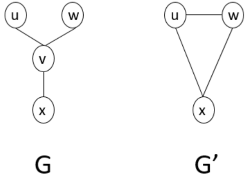
- A unique path from u to w through v

For each $(u, w) \in (E' \setminus E)$: $b'_{uw} \rightarrow b_{uv} \wedge b_{vw}$ (VE-4)

$$\sum_{(u,w)\in E' \setminus E} b'_{uw} \leq 1 \quad \text{(VE-5)}$$

Vertex Elimination Encoding for HCP (Cont.)

- Ensure the mapping: $H_{G'} \leftrightarrow H_G$



For each $(u, v) \in E$, $(v, w) \in E$, $u \neq w$:

$$b_{uv} \wedge b_{vw} \rightarrow b'_{uw} \qquad \text{(VE-6)}$$

$$b_{uv} \wedge b_{vw} \rightarrow \neg b_{uw} \qquad \text{(VE-7)}$$

For each $(u, v) \in E$, $(v, w) \in E$, $u \neq w$:

$$\neg b_{uv} \vee \neg b_{vw} \rightarrow b_{uw} = b'_{uw} \qquad \text{(VE-8)}$$

Vertex Elimination Encoding for HCP (Cont.)

- The correctness of VEE is guaranteed by the fact that a Hamiltonian cycle in G' corresponds to a Hamiltonian path from a neighbor w in $nbs^{+}(v)$ to a neighbor u in $nbs^{-}(v)$ of the eliminated vertex v $(u \neq w)$, and the path can be extended to a cycle by adding the arcs (u, v) and (v, w).
- Encoding size
  - $O(n^3)$ variables
  - $O(n^4)$ clauses.

Hybridize Distance and Vertex Elimination Encodings

- When the graph is sparse, use vertex elimination encoding
- When the graph is dense, use distance encoding
- Strategy used in the experiment:
  - if $d \times \sigma > n$, switch to distance encoding
    - d is the smallest degree
    - σ is the total number of eliminated vertices so far.

Implementation and Experimental Results

- Available in Picat version 3.7 with Kissat (picat-lang.org)
- Results on the Knight's Tour problem (seconds)

Instance	VEE	DIST	HYBRID
kt12	28.75	7.11	**0.32**
kt14	135.80	5.77	**1.23**
kt16	614.23	118.45	**2.72**
kt18	1050.80	16.55	**3.65**
kt20	TO	20.70	**6.16**
kt22	TO	19.60	**19.21**
kt24	TO	76.31	**46.03**
kt26	TO	TO	**116.14**
kt28	TO	TO	**192.73**
kt30	TO	TO	**200.98**

Experimental Results (Cont.)

- Results on Flinders instances (seconds)

Instance	VEE	DIST	HYBRID
graph162	TO	**33.89**	39.47
graph171	45.38	**5.35**	50.29
graph197	78.64	**13.16**	488.38
graph223	TO	**80.05**	200.71
graph237	125.66	**12.27**	237.51
graph249	62.48	**1.89**	61.04
graph252	182.27	**18.57**	468.85
graph254	**84.55**	TO	338.34
graph255	245.61	**31.30**	66.49
graph48	**0.75**	217.88	64.96

Experimental Results (Cont.)

- Knight's Tour 40×40
  - **DIST** fails to solve it in 24 hours!
  - **VEE** fails to translate the instance to CNF.
  - **HYBRID** solves it in 2711 seconds. → A big advance of the SOTA!

Conclusion

- Contributions
  - A working encoding based on vertex elimination for HCP
  - A hybrid encoding that combines VEE and distance encoding for HCP
  - Very encouraging results
- Further work
  - Is it possible to lower the encoding size of the basic VE encoding?
  - What is the best switching strategy?
  - What encodings dould be hybridized with VEE?

Thank you!

Questions?