On the complexity of integer programming with fixed-coefficient scaling

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supervised by Peter Jonsson

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LINKÖPINGS UNIVERSITET

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NP-complete in general

Can we simplify to something solvable in polynomial time?

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Idea: further restrict allowed inequalities

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Equations of the form $a_1x_1 \leq a_2x_2 + b$, $a_1, a_2 \in \mathbb{N}$, $b \in \mathbb{Z}$

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TVPI, restrict scaling coefficients to $\{\pm 1, \pm 2\}$ (BTVPI): In P (Wojciechowski & Subramani, 2023)

In this talk

TVPI, monotone inequalities, restrict scaling coefficients to powers of some $c \in \mathbb{N}$

Equations of the form $c^{a_1} x_1 \leq c^{a_2} x_2 + b$, $a_1, a_2 \in \mathbb{N}$, $b \in \mathbb{Z}$

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This is in P!

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WLOG, either a_1 or a_2 is zero

In this talk

TVPI, monotone inequalities, restrict scaling coefficients to powers of some $c \in \mathbb{N}$

Equations of the form $x_1 \leq c^a x_2 + b$ or $c^a x_1 + b \leq x_2$, $a \in \mathbb{N}$, $b \in \mathbb{Z}$

This is in P!

WLOG, either a_1 or a_2 is zero

Model as graph

A vertex per variable

A (directed) arc from x_1 to x_2 per inequality

Arcs have weight: $a_1 - a_2$

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Example:

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- $4x_2 + 1 \leq x_3$
- $x_3 \leq x_1 + 4$
- $x_3 \leq 4x_4$
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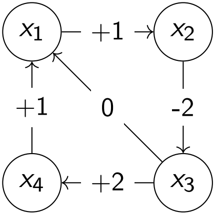

Transitioning into a finite domain

There exists an Ω such that:

If there exists a solution, then there exists one where each variable is bounded by Ω.

Furthermore, $\log(\Omega)$ is polynomial in the input size

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If there exists a solution, then there exists one where each variable is bounded by Ω.

Furthermore, $\log(\Omega)$ is polynomial in the input size

This is a (small variation of) a known result. (Papadimitriou, 1981)

Arc consistency

Associate a lower bound with each variable, initially $x_i \geq -\Omega$.

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If we reach $x_i > \Omega$ for some $i \implies$ No solution exists

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If no more propagations are possible $\implies$ Lower bounds form solution

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- $4x_2 + 1 \leq x_3$
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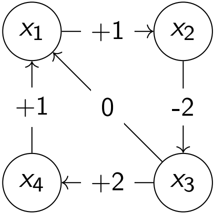
Suppose $\Omega = 10$.



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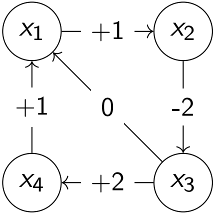

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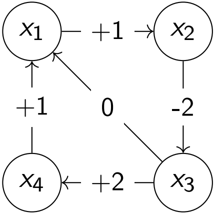

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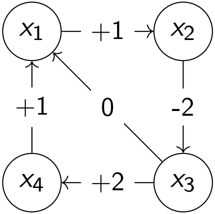

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$$x_3 \geq -10 \implies x_4 \geq -2 \implies x_1 \geq 0$$

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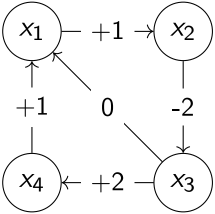

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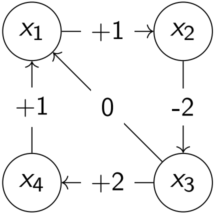

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Problem: the propagation chain gets way too long ($\mathcal{O}(\Omega n)$)

Edge shortening

Consider the first two steps of the chain
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If we add this arc: $x_3 \geq -10 \implies x_1 \geq 0$

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Solution: preprocess graph, add new arc for every two consecutive arcs

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For any three consecutive arcs, we can combine at least one pair

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New problem: way too many new arcs. Coefficients get way too large.

Edge compression

We can remove irrelevant arcs. We can replace arcs with large coefficients with arcs with lower coefficients.

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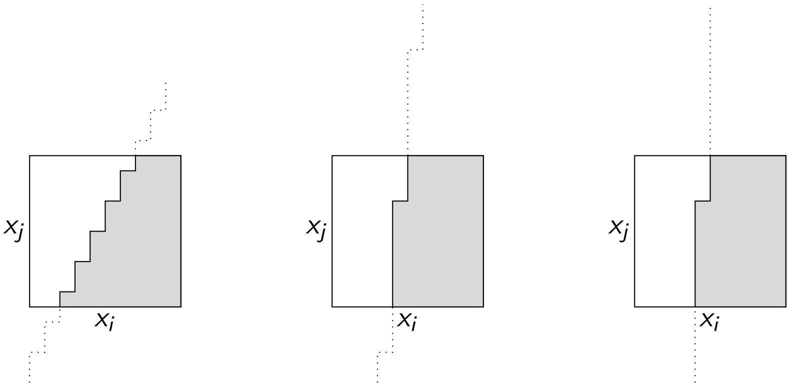
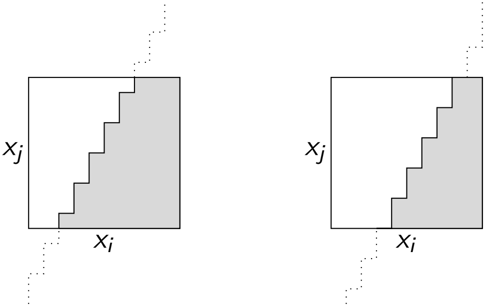
Consider arcs from x_i to x_j.



Redundant arcs

Observation: every arc is equivalent to one with weight at most $\mathcal{O}(\log(\Omega))$

Now consider arcs from x_i to x_j with the same weight.



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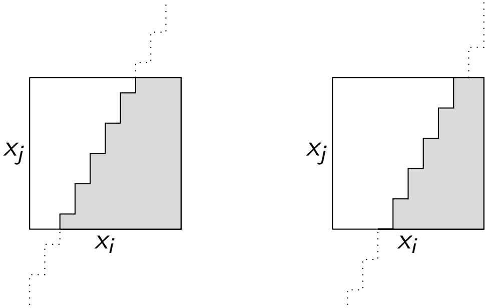

We only need the arc with the most restrictive constant term

Arc count

Overall, we only need $\mathcal{O}(\log(\Omega))$ arcs between any two vertices

This means at most $\mathcal{O}(n^2 \log(\Omega))$ arcs in total.

Final algorithm

- transition into finite domain: compute Ω
- repeat $\mathcal{O}(\log(\Omega))$ times:
  - merge consecutive arcs into new arcs (if possible)
  - resize arcs with too large coefficients
  - remove irrelevant arcs
- enforce arc consistency

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