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Strengthening Relaxed Decision Diagrams for Maximum Independent Set Problem: Novel Variable Ordering and Merge Heuristics

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NRW.

Introduction

Decision Diagrams (Exact)

- a **graphical representation of the solution space**
- **generic**: require a **Dynamic Programming** formulation
- **exponential size**

Approximate Decision Diagrams (restricted and **relaxed)**

- **linear size**
- obtain (combinatorial) **primal and dual bounds**
- put in **generic exact algorithms (e.g. B&B** or **P&B)**
- **quality of the bounds depend on heuristic strategies**

  for **1. node selection** and **2. variable ordering**

Maximum Independent Set Problem (MISP)

- **NP-hard** combinatorial problem
- Problem: Given a graph $G = (V, E)$, find the largest $I \in V$ subset of its vertices such that $\forall\, v, u \in I, (u, v) \notin E$

- IP formulation:
- $Maximize \quad \sum_{v \in V} x_v$

  S.t. $\quad x_u + x_v \leq 1, \forall\, (u, v) \in E$

  $x_v \in \{0,1\}$

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Learning more about Decision Diagrams for Optimization

- **A gentle introduction:**

TUTORIALS IN OPERATIONS RESEARCH
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An Introduction to Decision Diagrams for Optimization

Willem-Jan van Hoeve
Tepper School of Business, Carnegie Mellon University, vanhoeve@andrew.cmu.edu

- **A survey:**

Home > INFORMS Journal on Computing > Vol. 34, No. 4 >

Decision Diagrams for Discrete Optimization: A Survey of Recent Advances

Margarita P. Castro, Andre A. Cire, J. Christopher Beck

Published Online: 22 Mar 2022 | https://doi.org/10.1287/ijoc.2022.1170

Some Decision Diagram based Solvers

DDO, a Generic and Efficient Framework for MDD-based Optimization

By Xavier Gillard, Pierre Schaus, Vianney Coppé

Developed in Rust

Peal-and-Bound: Solving Discrete Optimization Problems with Decision Diagrams and Separation

PhD Thesis by Isaac Rudich (August 2024)

Developed in Julia

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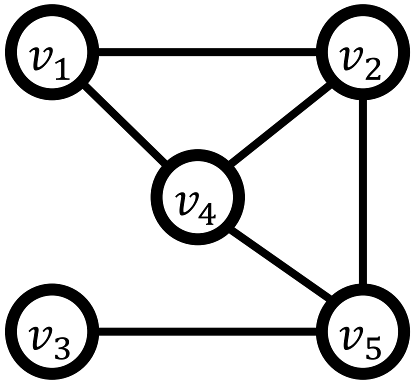
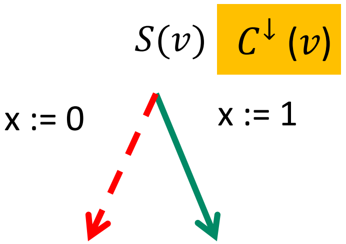
Top-Down Compilation of Exact DD for MISP





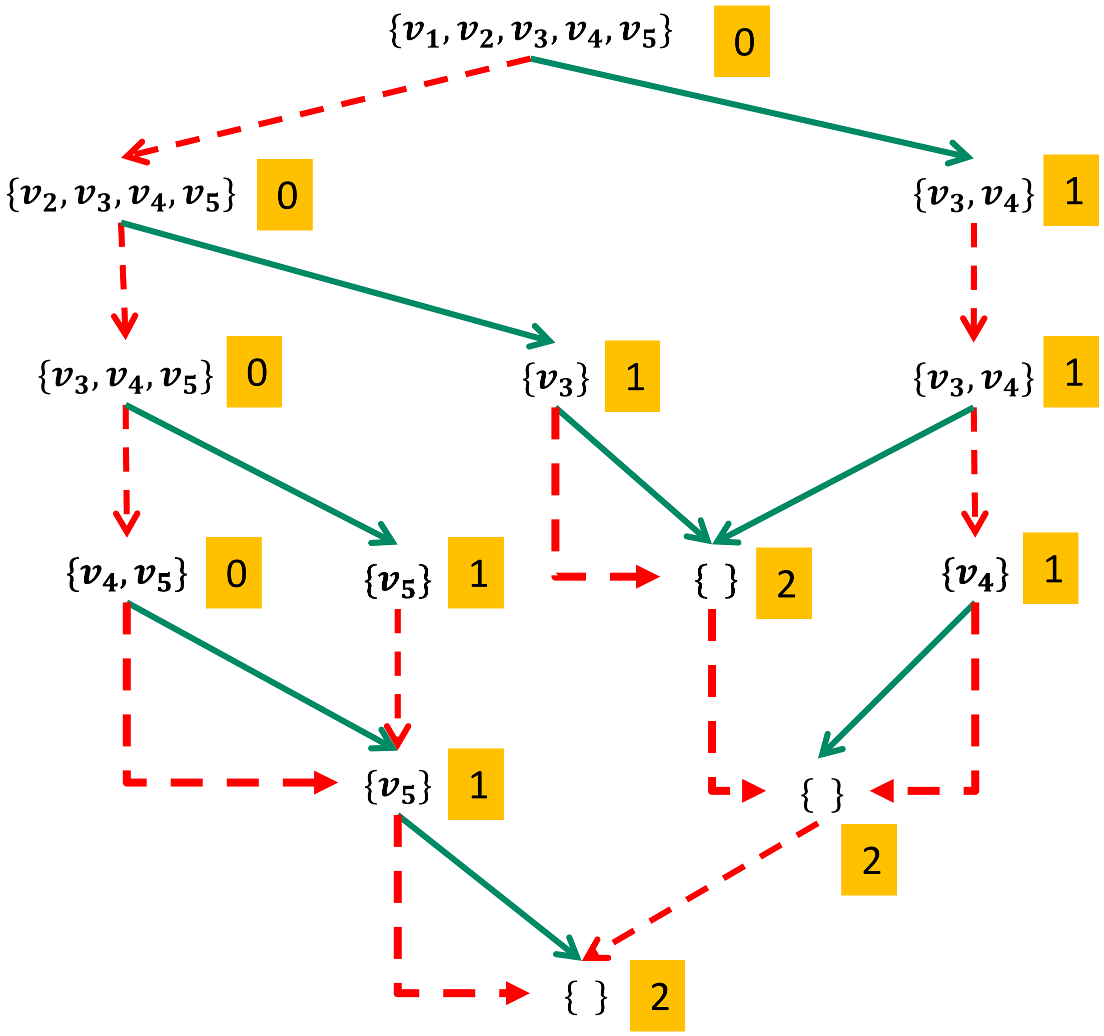
$x_1 = v_1$

$x_2 = v_2$

$x_3 = v_3$

$x_4 = v_4$

$x_5 = v_5$



For the MISP:
State S: Admissible vertices
State transition $f(S, x_i)$:

$$f(S, x_i) = \begin{cases} S \text{ for } x_i = 0 \\ S \setminus N(x_i) \text{ for } x_i = 1 \end{cases}$$

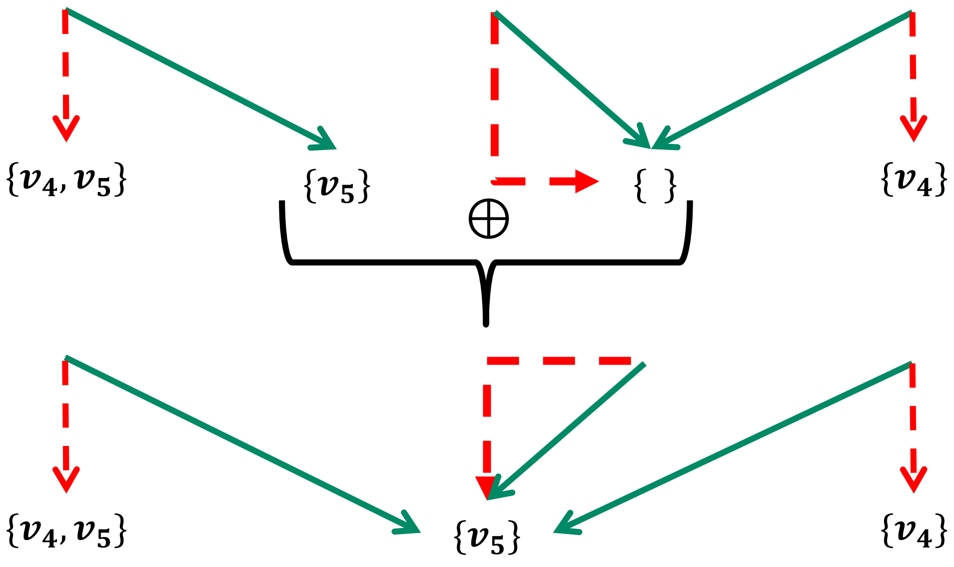
Top-Down Compilation of Relaxed DD for MISP

Dual Bounds via Relaxed Decision Diagrams

- given a maximum permitted width W
- if the width of a layer exceeds W: **merge nodes** until W is respected
- to obtain the state of the merged nodes, apply a problem-specific merge operation ($\oplus$) that ensures a **valid relaxation**:

For MISP: $\oplus = \cup$

$i.e.\ S_1 \oplus S_2 \oplus \cdots \oplus S_k = S_1 \cup S_2 \cup \cdots \cup S_k$



Merging nodes introduces errors

The error (hence quality of the bounds) depends on:

1. node selection, 2. variable ordering

How can we control (decrease) the error?

Some Related Works on Variable Ordering

Using Machine Learning

- using Reinforcement Learning (Cappart et. al 2019, 2021, 2023) [MISP]
- Algorithm selection (Karahalios and van Hoeve 2021)

Classical methods (no maching learning)

- generic: look-ahead (Bergman et al. 2016, book)
- specific [MISP]: state information, i.e. MIN (Bergman et al. 2016, book)

This paper:

→use **graph-theoretical** information from **induced subgraphs corresponding to states**

Some Related Works on Node Selection

Strategies based on Sorting

(keep the best W-1 nodes, merge the rest)

- by objective function value (SO)
- using state information
- using binary classifiers, look-ahead (Frohner and Raidl 2019) [MISP]
- tie-breaking strategies for nodes with identical values (Frohner and Raidl 2020) [MISP]

Strategies based on Grouping

(group nodes according to state similarity)

- using labeling functions (Horn et al. 2021)
- using problem-specific distances (de Weerdt et al. 2021)
- using unsupervised clustering (Nafar and Römer 2024)
- approximate equivalence using lookahead, merge and reduce (Nafar and Römer 2024)

This paper:

→use **Border Tie** (BT) merging for improving the **node selection**

Errors in relaxed DDs for MISP

- Errors in a merged node is a **“two-sided” over-approximation** of all the nodes in the merge:

forming a super-set of their states

↓

over-approximating the states of the merged nodes

↓

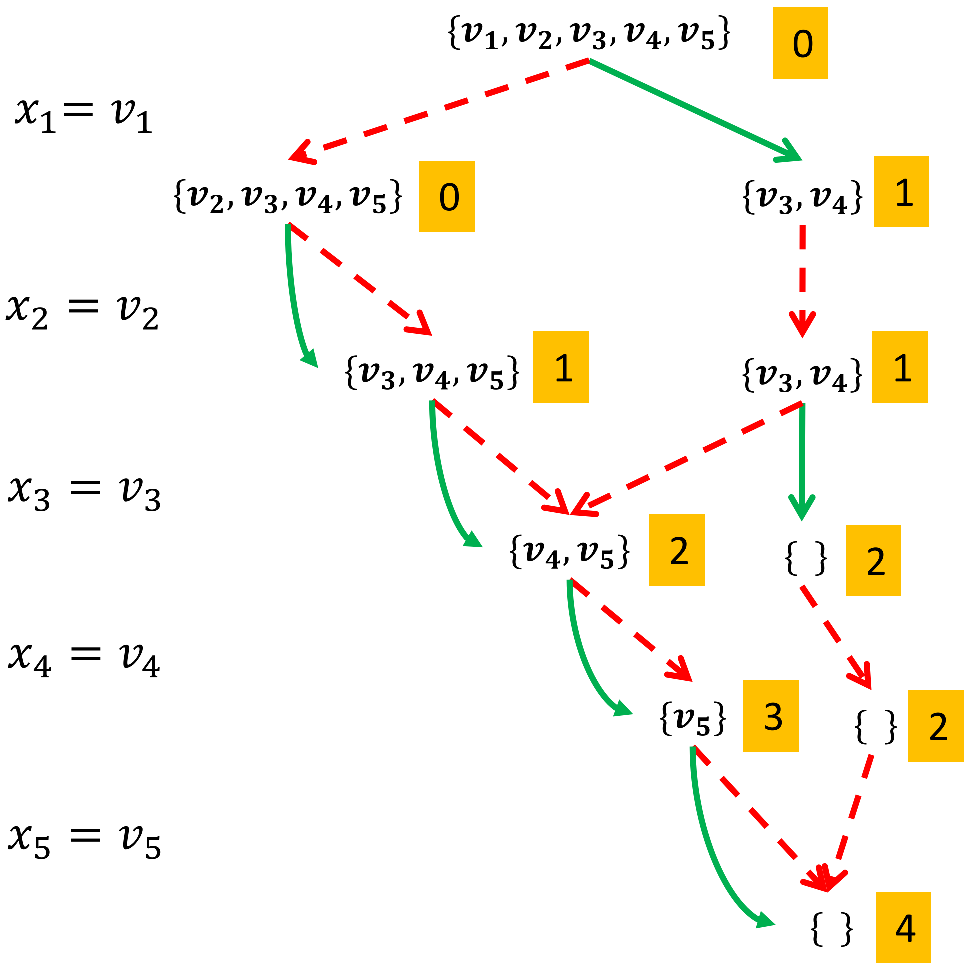
less diversity of states

↓

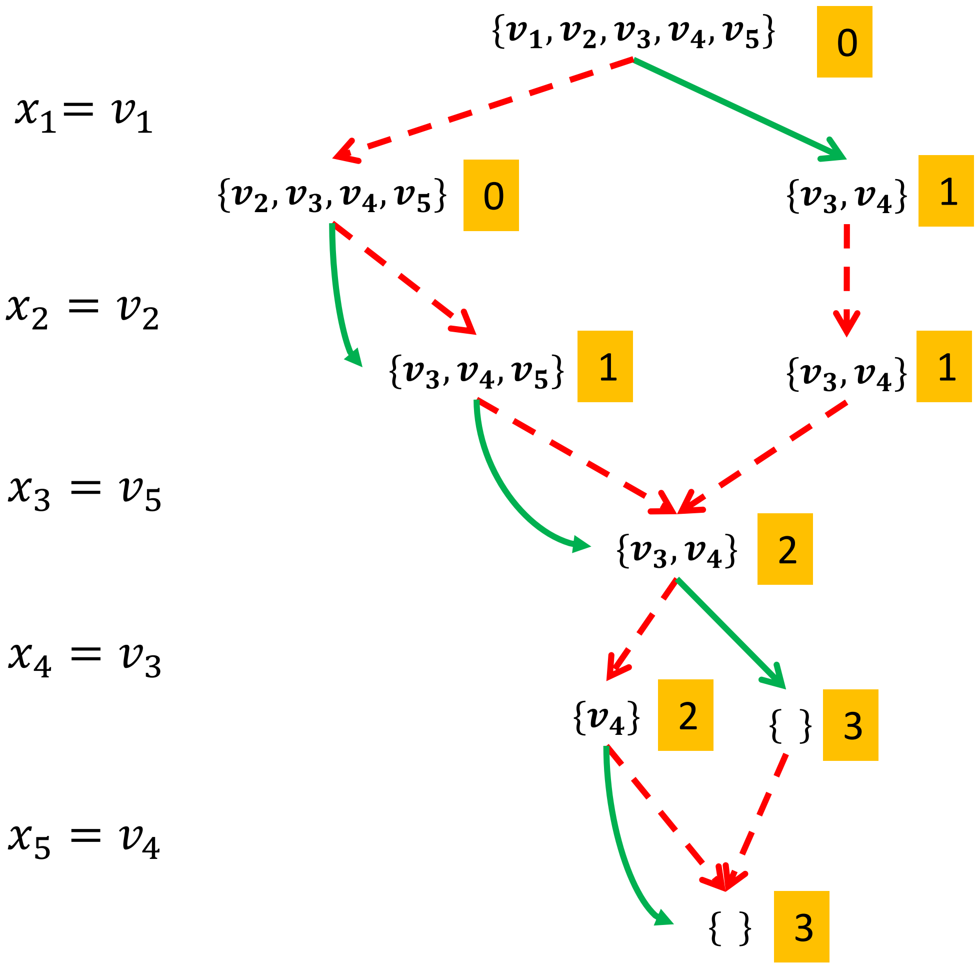
Variable Ordering (Dynamic)

Relaxed DDs for MISP instance

- no particular variable ordering



- **MIN** variable ordering



Current Degree Sum: CDS Variable Ordering

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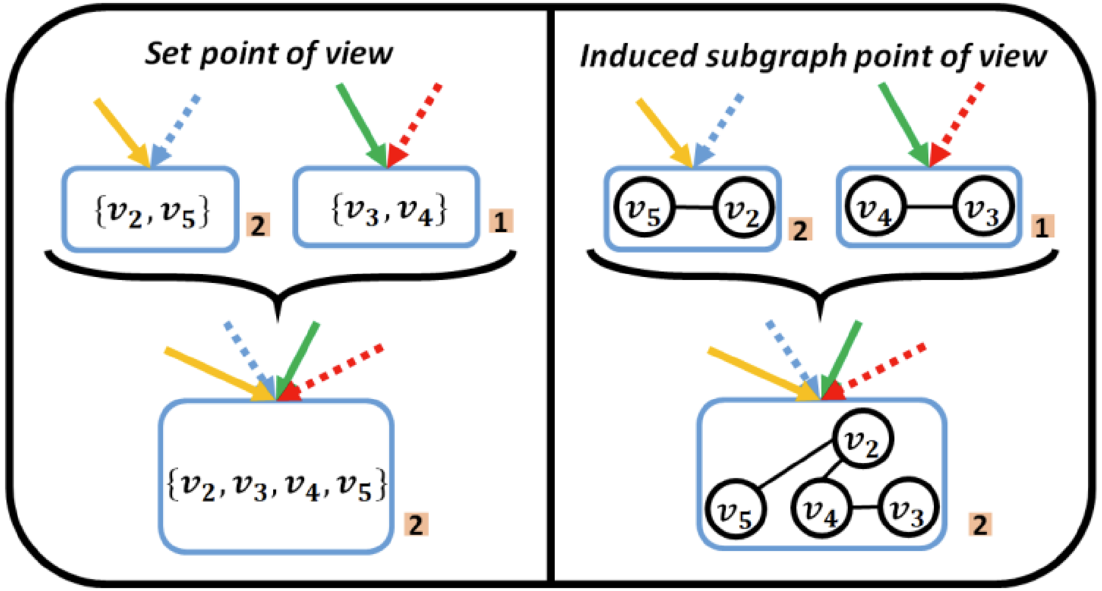
- Change the **point of view:** Original problem
- Original problem: More information



1. Less Diversity (similar states)
2. More intersection with optimum solutions
3. Potentially smaller layers

But layers have more than one graph!!!

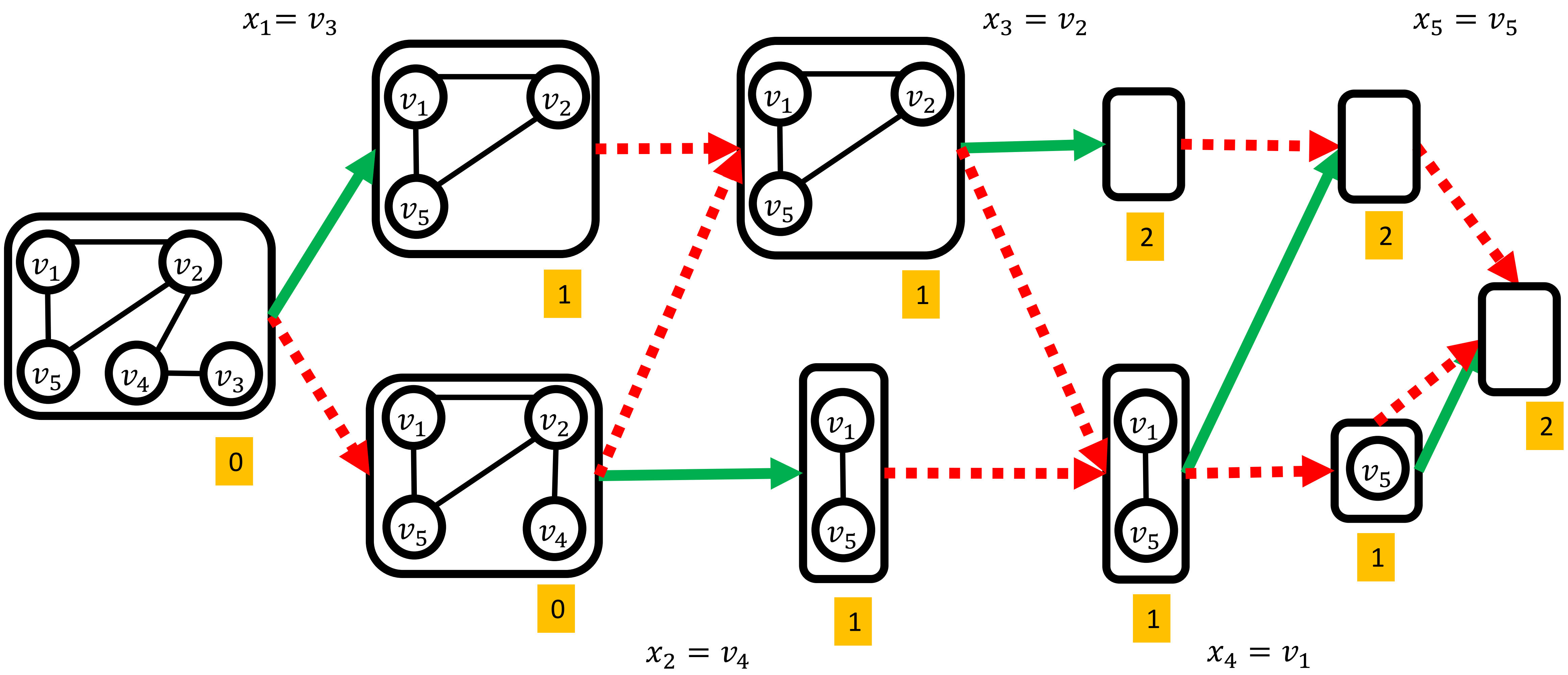
HOW?!

Vertex with lower degree, if one graph (state)

Current Degree Sum: CDS

$$\sum_{s \in L} dg_s(v_i)$$

Relaxed DD for MISP using CDS variable ordering



Errors in relaxed DDs for MISP

- Errors in a merged node is a **“two-sided” over-approximation** of all the nodes in the merge:

forming a super-set of their states → over-approximating the states of the merged nodes → less diversity of states → **Variable Ordering (Dynamic)**

redirecting the in-arcs → over-approximating the longest paths from root → closer objective values → **Node Selection (Tie-Based Merge)**

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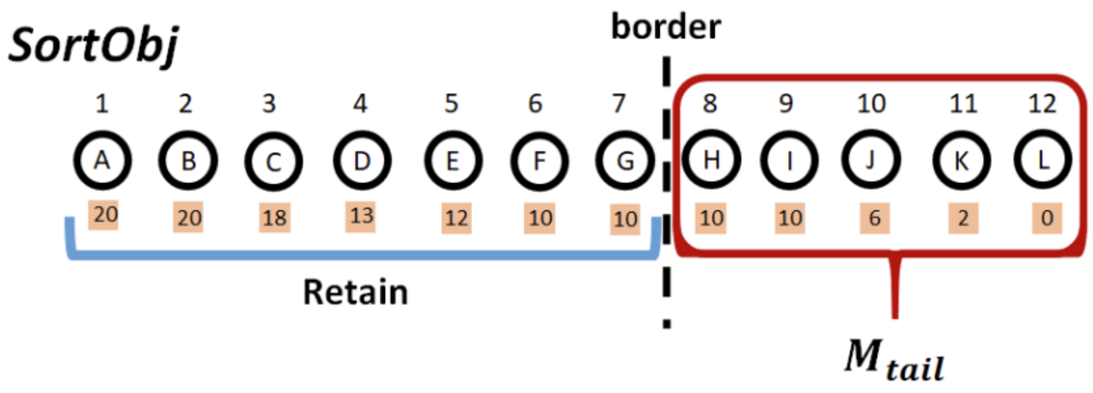
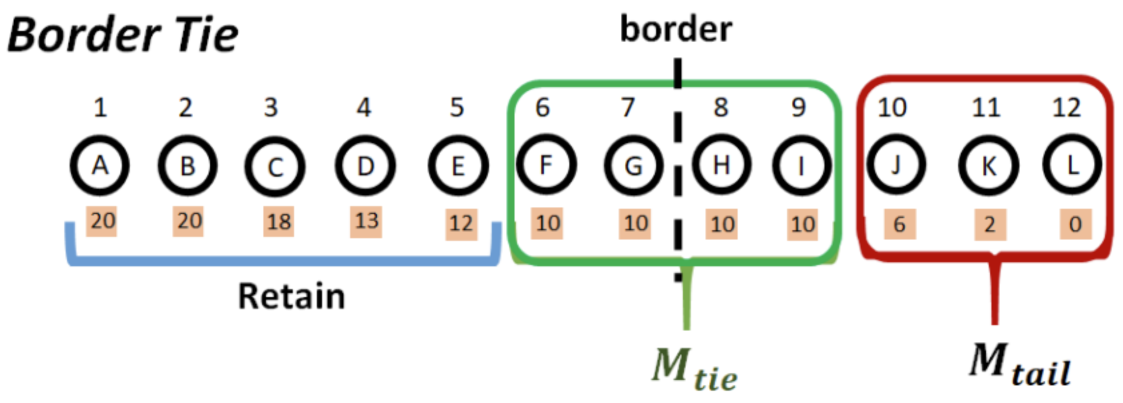
Node Selection in relaxed DDs for MISP

Definition 1. (Border Tie). *Let the nodes in a layer of a DD be sorted according to some criterion C and SortC be its corresponding sorted list, and let W be the given maximum width. A subsequence of SortC in which all nodes have the same criterion value and which includes SortC [W − 1] and SortC [W] is called a Border Tie.*

1. keep the best W-1 nodes according to objective value
2. merge the rest



1. keep the best W-1 nodes according to objective value
2. merge border tie nodes (if any)
3. merge the rest (if any)



Computational Results: Gap-Time (Dual bounds)

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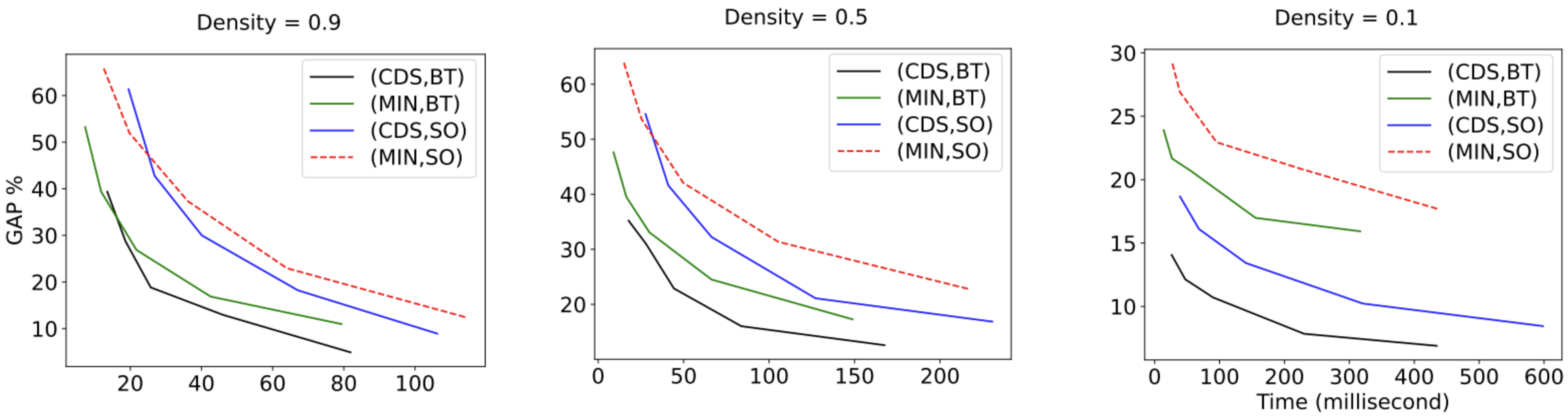
- random graph instances, 20 instance per density each with 100 vertices
- Comparing the bounds obtained with our approaches to those with the standard approaches



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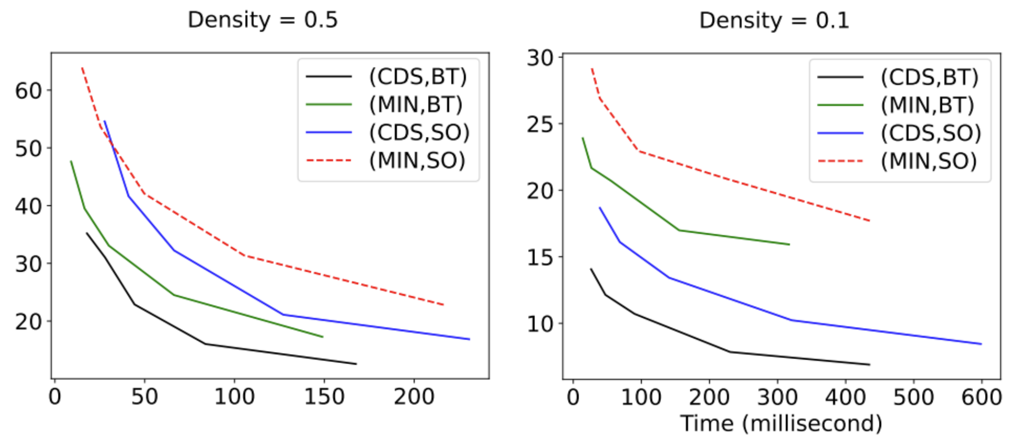
Computational Results: DD Sizes



Density	Average DD Size (nodes)			
	(MIN,SO)	(**CDS**,SO)	(MIN,**BT**)	(**CDS**,**BT**)
0.1	73513	76190	**58059**	59069
0.2	71324	73190	**56393**	57173
0.3	68484	69470	**54020**	55253
0.4	65088	66580	**51870**	53049
0.5	62053	63355	**50456**	51356
0.6	58946	59661	**47715**	47505
0.7	55058	56040	**43606**	45629
0.8	50808	51466	**35334**	36025
0.9	45770	47272	**37578**	41607

- SO vs **BT**: smaller DD + more merged nodes, but better dual bounds: ***Success?!***
- MIN vs **CDS**: control the diversity of the states to reduce the destructive effect of merge: ***Success?!***

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Computational Results: Branch-and-Bound

Density	Time (Seconds)			
	(MIN,SO)	(**CDS**,SO)	(MIN,**BT**)	(**CDS**,**BT**)
0.1	872.3	821.9	445.8	**320.9**
0.2	126.4	94.0	85.5	**56.4**
0.3	31.0	26.8	24.3	**17.0**
0.4	10.1	9.3	8.6	**5.9**
0.5	4.3	3.5	4.0	**2.5**
0.6	1.92	1.70	1.57	**1.4**
0.7	1.11	1.01	0.93	**0.89**
0.8	0.73	0.70	0.81	**0.66**
0.9	0.60	**0.56**	0.65	0.61

- Any combination containing **BT** or **CDS** or both is better than (MIN,SO)

Density	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	Average
(**CDS**,SO)	5.7	25.61	13.42	8.4	18.3	11.4	9.0	4.1	**6.6**	11.43
(MIN,**BT**)	48.8	32.3	21.4	15.6	6.2	18.2	16.2	-10.9	-8.3	15.52
(**CDS**,**BT**)	**63.2**	**55.3**	**45.2**	**42.0**	**40.6**	**25.52**	**19.8**	**9.5**	-1.6	**33.30**

Computational Results: Branch-and-Bound

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(MIN,**BT**) better **bounds** and **solution times** than (**CDS**,SO), but more subproblems to solve: **good quality bounds is not the only factor?!**

Hypothesis: the intuitions behind the design of **CDS** (more intersections with optimal solutions)

Density	Node Size (Sub-problem Solved in B&B)			
	(MIN,SO)	(**CDS**,SO)	(MIN,**BT**)	(**CDS**,**BT**)
0.1	63880	39051	34183	**17150**
0.2	15454	6623	9088	**3816**
0.3	5906	3053	3830	**1700**
0.4	2609	1581	1814	**802**
0.5	1452	796	947	**450**
0.6	702	397	440	**266**
0.7	399	256	283	**195**
0.8	291	210	245	**173**
0.9	247	**174**	224	178

(**CDS,BT)**: benefits of both, (good bounds, more intersections): solved subproblems reduced by 50%

Density	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	Average
(**CDS**,SO)	38.8	57.1	48.3	39.4	45.2	43.3	35.7	27.7	29.6	40.6
(MIN,**BT**)	46.4	41.1	35.1	30.4	34.7	37.3	29.0	15.8	9.3	31.0
(**CDS**,**BT**)	**73.1**	**75.30**	**71.2**	**69.2**	**68.9**	**62.0**	**51.0**	**40.3**	**27.6**	**59.8**

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Conclusions

- We proposed for MISP:
    - a novel variable ordering approach which is based-on graph theoretical information obtained from induced subgraphs corresponding to states
    - a new tie-based node selection approach
- These simple ideas yield considerably stronger dual bounds than the standard and most commonly used approaches
- As was shown in multiple previous papers, better bounds translate into faster exact algorithms such as branch-and-bound

References

Using Clustering to Strengthen Decision Diagram Bounds for Discrete Optimization.
AAAI 2024

Lookahead, Merge and Reduce for Compiling Relaxed Decision Diagrams for Optimization.
CPAIOR 2024

Strengthening Relaxed Decision Diagrams for Maximum Independent Set Problem: Novel Variable Ordering and Merge Heuristics.
CP 2024