The complexity of symmetry breaking beyond lex-leader

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Symmetries in SAT

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Example

$F = (x_1 \lor x_2 \lor \neg x_3) \land (x_2 \lor x_3 \lor \neg x_1) \land (x_1 \lor x_3 \lor \neg x_2)$

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Output: Does F have a satisfying assignment?

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- θ, θ' are called symmetric if $\theta' = \varphi(\theta)$ for $\varphi \in \mathrm{Aut}(F)$

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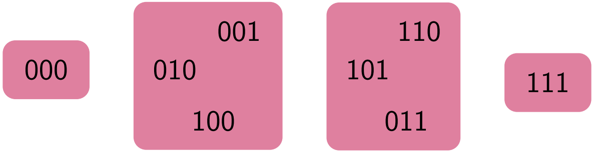

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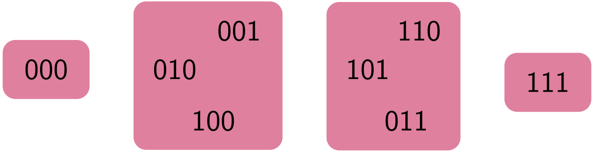

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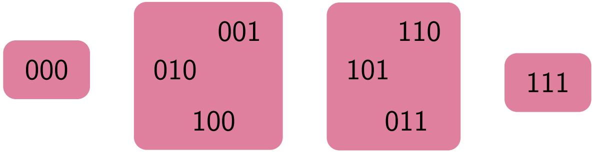

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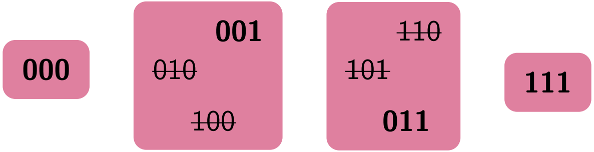

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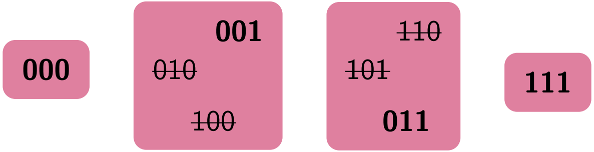

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Static symmetry breaking: add clauses to falsify all but one assignment per orbit

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Problem (Symmetry breaking)

Given $G \leq \mathrm{Sym}(x_1, \ldots, x_n)$, compute a symmetry breaking CNF predicate for G.

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Weaker variant

Given $G \leq \mathrm{Sym}(x_1, \ldots, x_n)$, compute a symmetry breaking circuit for G.

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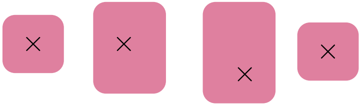
Weaker variant

Given $G \leq \mathrm{Sym}(x_1, \ldots, x_n)$, compute a symmetry breaking circuit for G.

- the circuit is even allowed to introduce additional variables

Lex-leader constraints

Idea: compute a CNF predicate only true of the lex-leader in each orbit of symmetric assignments



- in practice: incomplete lex-leader constraints

Theorem (Crawford et. al., 1997)

Computing for any formula F a predicate only true of the lex-leader in each orbit of symmetric assignments is NP-hard.

This also holds for

- restricted classes of groups [Luks-Roy, 2004]
- orders similar to lex-leader [Katsirelos et. al. 2010, Walsh 2020+]

Beyond lex-leader

- wreath symmetries using the global cardinality constraint together with lex-leader constraints [Flener et. al. 2009]
- constraints for graph problems similar to NAUTY [Codish et. al. 2016]
- minimal SAT symmetry breaking constraints for small groups [Heule 2019]

Goal

Complexity classification for computing symmetry breaking predicates

- for groups, and
- independently of the method to choose representatives.

Questions

- How does symmetry breaking relate to problems such as graph isomorphism?
- Find a general strategy to show hardness.

Complexity landscape of graph isomorphism

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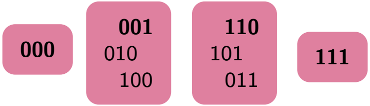
Given $G \leq \mathrm{Sym}(n)$ and $\sigma \in \{0,1\}^n$, find a canonical representative in the orbit of σ (consistent among all possible inputs).



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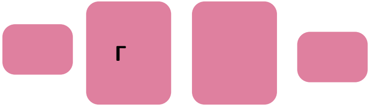

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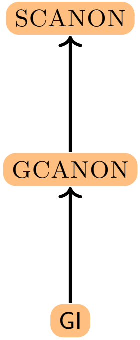

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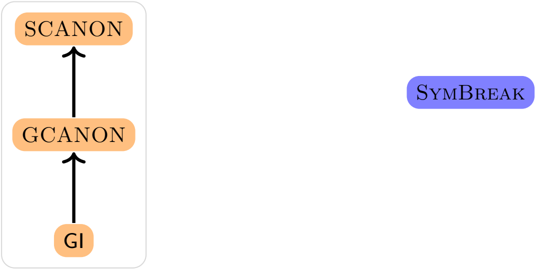

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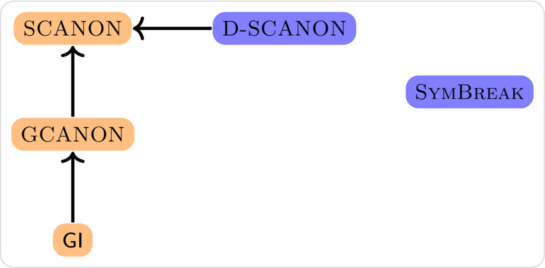

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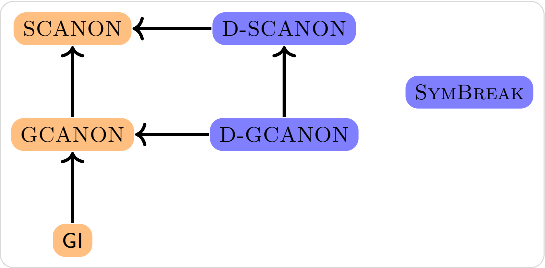

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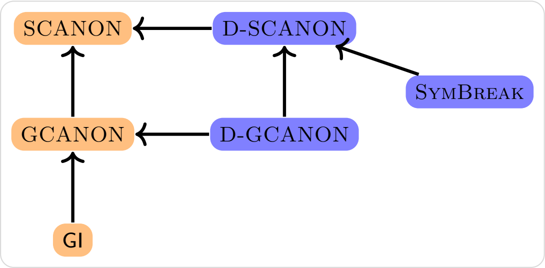
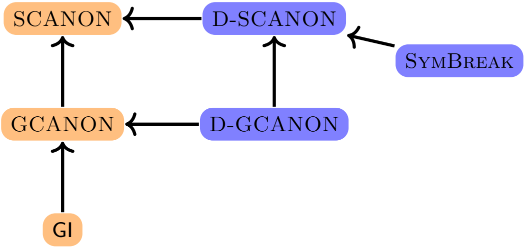


Quasipolynomial bound

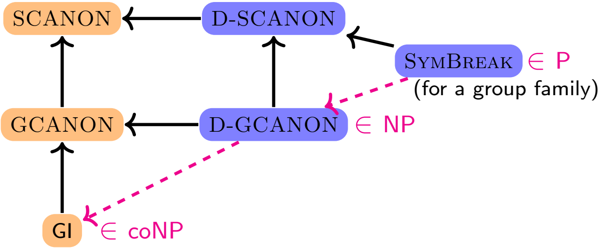


Theorem (Babai, 2016)

SCANON can be solved in quasipolynomial time.

Consequence: use this there is a quasipolynomial time algorithm to produce a symmetry breaking circuit of quasipolynomial size

Barriers for symmetry breaking

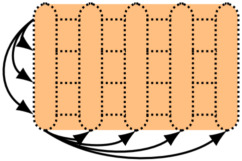


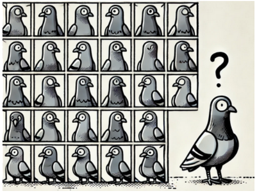
Theorem (Anders-B.-Rattan, 2024)

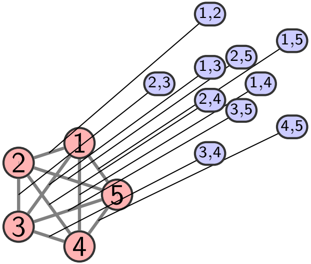
D-GCANON ∈ NP implies GI ∈ coNP. Which is a major unsolved problem!

Strategy for showing hardness: given a symmetry breaking circuit for a group family, use it to show D-GCANON ∈ NP.

Row-column symmetries and Johnson symmetries



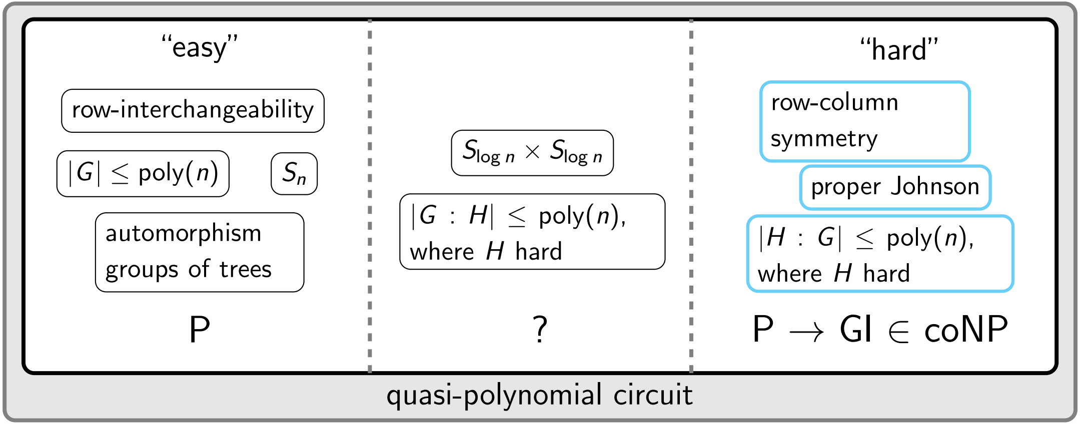




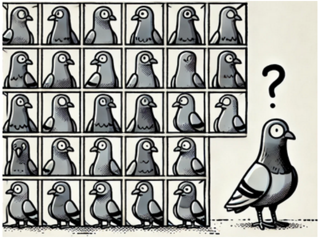
Theorem (Anders-B.-Rattan, 2024)

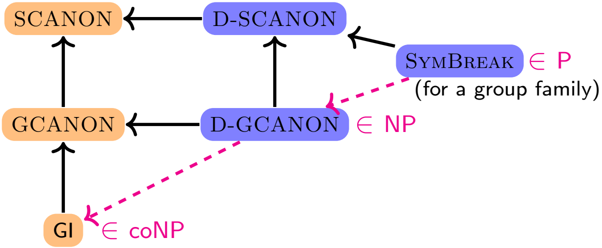
Suppose that a symmetry breaking circuit for row-column symmetries or proper Johnson symmetries can be efficiently computed. Then GI $\in$ coNP.

Outlook: group-based complexity classification



Muchas gracias - thank you!





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https://github.com/markusa4/satsuma